South East Asian J. of Mathematics and Mathematical Sciences
Vol. 22, No. 1 (2026), pp. 109-124

DOI: 10.56827/SEAJMMS.2026.2201.8 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

HERMITE BASED UNIFIED APOSTOL-BERNOULLI-EULER POLYNOMIALS

Nabiullah Khan, Mohd Zeeshan, Owais Khan* and Nafis Ahmad**

Department of Applied Mathematics,
Faculty of Engineering and Technology,
Aligarh Muslim University, Aligarh - 202002, (U.P.), INDIA

E-mail : nukhanmath@gmail.com, mzmath04@gmail.com

*Department of Mathematics and Statistics,
Integral University, Lucknow - 226026, (U.P.), INDIA

E-mail : owkhan5@gmail.com

**Department of Mathematics,
Shibli National Collage, Azamgarh - 276001, (U.P.), INDIA

E-mail : nafis.sncmaths@gmail.com

(**Received:** *Jun. 15, 2025* **Accepted:** *Apr. 06, 2026* **Published:** *Apr. 30, 2026)*

Abstract: In this paper, we present a novel category of generalized Hermite based unified Apostol-Bernoulli-Euler polynomials and drive several implicit summation formulas through generating functions. These results extend some known summations and identities of unified Bernoulli-Euler polynomials of Apostol type, generalized Hermite-Bernoulli polynomials, and generalized Hermite-Euler polynomial.

Keywords and Phrases: Bernoulli polynomials, Euler polynomials, Hermite-Bernoulli polynomial of Apostol type, Hermite Euler polynomial of Apostol type, Hermite Bernoulli polynomial, Hermite Euler polynomial.

2020 Mathematics Subject Classification: 11B68, 33C10, 33C45, 33E20.

1. Introduction

The specialized functions of mathematical physics have experienced notable advancement in recent years, particularly in their generalized and multi variable forms. Specifically, the introduction of special polynomials in two variables has offered fresh analytical approaches for resolving extensive arrays of partial differential equations frequently encountered in physical contexts. The genesis of many of these specialized functions and their extensions can be attributed to the challenges posed by various physical phenomena. Throughout this paper, the symbol $\mathbb{C}, \mathbb{R}$ and $\mathbb{Z}$ denote the sets of complex numbers, real numbers, and integers, respectively.

The Bernoulli polynomials $\{\mathcal{B}_n(\rho)\}_{n\geq 0}$, originally introduced by Bernoulli, J. [3], and the Euler polynomials $\{\mathcal{E}_n(\rho)\}_{n\geq 0}$, introduced by Euler, L. [9], rank among the most vital families of special polynomials, with wide applications in mathematical analysis and number theory. These are typically introduced through their generating functions:

$$\frac{te^{\rho t}}{e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(\rho) \frac{t^n}{n!} \qquad (|t| < 2\pi), \tag{1.1}$$

and

$$\frac{2e^{\rho t}}{e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(\rho) \frac{t^n}{n!} \qquad (|t| < \pi). \tag{1.2}$$

In particular, for $\rho = 0$, the quantities $\mathcal{B}_n = \mathcal{B}_n(0)$ and $\mathcal{E}_n = \mathcal{E}_n(0)$ are called the Bernoulli numbers and Euler numbers, respectively. These polynomials and numbers play significant roles in various branches of mathematics, including number theory, combinatorics, numerical analysis, and the theory of special functions.

A natural and important generalization of the Bernoulli polynomials was introduced by Apostol, T. M. [1], leading to so-called Apostol-Bernoulli polynomials $\{\mathcal{B}_n(\rho, \tau)\}_{n\geq 0}$ defined by the generating functions

$$\frac{te^{\rho t}}{\tau e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(\rho, \tau) \frac{t^n}{n!} \qquad (|t + ln\tau| < 2\pi, \tau \in \mathbb{C} \setminus \{0\}). \tag{1.3}$$

Similarly, the Apostol-Euler polynomials $\{\mathcal{E}_n(\rho, \tau)\}_{n\geq 0}$, introduced by Srivastava, H. M. [1], are generated by:

$$\frac{2e^{\rho t}}{\tau e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(\rho, \tau) \frac{t^n}{n!} \qquad (|t + ln\tau| < \pi, \tau \in \mathbb{C} \setminus \{0\}). \tag{1.4}$$

These extensions reduce to the classical Bernoulli and Euler polynomials when $\tau = 1$ and have been widely studies due to their rich analytic and combinatorial properties (see, [12]) Luo, Q. M. and Srivastava, H. M. [11], introduced the generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\kappa)}(\rho,\tau)$ of order $\kappa \in \mathbb{C}$. Luo, Q. M. [13], proposed the generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\kappa)}(\rho,\tau)$ of order $\kappa \in \mathbb{C}$. These polynomials are defined as follows, respectively.

Definition 1.1. *For $\tau \in \mathbb{C} \setminus \{0\}$, the generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\kappa)}(\rho,\tau)$ of order κ are commonly defined using the following generating function:*

$$\left(\frac{t}{\tau e^t - 1}\right)^{\kappa} e^{\rho t} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\kappa)}(\rho,\tau)\frac{t^n}{n!}, \quad (|t + ln\tau| < 2\pi; 1^{\kappa} := 1), \tag{1.5}$$

with

$$\mathcal{B}_n^{(\kappa)}(0,\tau) = \mathcal{B}_n^{(\kappa)}(\tau).$$

Which is known as Apostol-Bernoulli numbers $\mathcal{B}_n^{(\kappa)}(\tau)$ of order κ. The generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\kappa)}(\rho,\tau)$ of order κ defined by the following generating function:

$$\left(\frac{2}{\tau e^t + 1}\right)^{\kappa} e^{\rho t} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\kappa)}(\rho,\tau)\frac{t^n}{n!}\ , \quad (|t + ln\tau| < \pi; 1^{\kappa} := 1) \tag{1.6}$$

with

$$\mathcal{E}_n^{(\kappa)}(0,\tau) = \mathcal{E}_n^{(\kappa)}(\tau),$$

which is known as Apostol-Euler numbers of order κ.

In addition to these families Hermite polynomial $\{\mathcal{H}_n(\rho)\}_{n\geq 0}$ constitute another important class of special polynomials. They are defined by the generating function [18]:

$$e^{2\rho t - t^2} = \sum_{n=0}^{\infty} \mathcal{H}_n(\rho)\frac{t^n}{n!}. \tag{1.7}$$

A natural generalization was introduced by Appell, P. and Kampe de Feriet, J. [2], through the following generating function :

$$e^{\rho t + \sigma t^2} = \sum_{n=0}^{\infty} \mathcal{H}_n(\rho,\sigma)\frac{t^n}{n!}. \tag{1.8}$$

In (2012), Pathan, M. A. [15] introduced the generalized Hermite-Bernoulli polynomials ${}_H\mathcal{B}_n^{(\kappa)}(\rho,\sigma)$ of order κ, using following generating function:

$$\left(\frac{t}{e^t - 1}\right)^{\kappa} e^{\rho t + \sigma t^2} = \sum_{n=0}^{\infty} {}_H\mathcal{B}_n^{(\kappa)}(\rho,\sigma)\frac{t^n}{n!}, \quad (|t + ln\tau| < 2\pi; 1^{\kappa} := 1), \tag{1.9}$$

it is a generalization of well-known result of Dattoli, G., Lorenzutta, S. and Cesarano, C. ([8], p.386 (1.6)).

$$\frac{t}{e^t-1}e^{\rho t+\sigma t^2}=\sum_{n=0}^{\infty}{}_H\mathcal{B}_n(\rho,\sigma)\frac{t^n}{n!}. \tag{1.10}$$

Also in (2014), Pathan, M. A. and Khan, W. A. [17] introduced the generalized Hermite-Euler polynomials ${}_H\mathcal{E}_n^{(\kappa)}(\rho;\sigma)$ of order κ which are given by the following generating function:

$$\left(\frac{2}{e^t+1}\right)^{\kappa}e^{\rho t+\sigma t^2}=\sum_{n=0}^{\infty}{}_H\mathcal{E}_n^{(\kappa)}(\rho,\sigma)\frac{t^n}{n!},\quad (|t+ln\tau|<\pi;1^{\kappa}:=1). \tag{1.11}$$

In (2021), Belbachir, H., Djemmada, Y. and Hadj-Brahim, S. [6], introduced the unified Bernoulli-Euler polynomials of Apostol type as determined by the power series for $\tau\in\mathbb{R}_+^*$ and $\xi\in\mathbb{R}_+-\{1\}$:

$$\left[\frac{2-\xi+\frac{\xi t}{2}}{\tau e^t+(1-\xi)}\right]e^{\rho t}=\sum_{n=0}^{\infty}\mathcal{B}_n(\rho,\tau,\xi)\frac{t^n}{n!}, \tag{1.12}$$

where

$$\begin{cases}\left|ln\left(\frac{\tau}{1-\xi}+t\right)\right|<2\pi, & \text{for}\quad 0\le\xi<1\\[2ex] \left|ln\left(\frac{\tau}{\xi-1}+t\right)\right|<\pi, & \text{otherwise.}\end{cases}$$

Motivated by their significance and applicability in various domains such as number theory, combinatorics, classical and numerical analysis, and physics, numerous categories of generalized Bernoulli-Euler numbers and polynomials, along with generalized Hermite-Bernoulli and Hermite-Euler polynomials, Bell based Euler polynomials have recently garnered attention from multiple researchers (see, [14]). We introduce in this paper, a new class of generalized Hermite based unified Apostol-Bernoulli-Euler polynomials ${}_H\beta_n(\rho,\sigma,\tau,\xi)$. The aim of this study is to provide a comprehensive overview of these families in a unified and generalized manner. We establish fundamental properties and derive implicit summation formulas for the Hermite based unified Apostol-Bernoulli-Euler polynomials utilizing various analytical techniques applied to their respective generating functions.

2. Hermite based unified Apostol-Bernoulli-Euler polynomials

In this section, we introduce the Hermite based unified Apostol-Bernoulli-Euler polynomials and compile a table that signifies the relation of these polynomials with certain other interesting polynomials. Furthermore, we established determinantal representation of these polynomials.

Definition 2.1. *Let $\tau \in \mathbb{R}_+^*$ and $\xi \in \mathbb{R}_+ - \{1\}$, the following power series defines the Hermite based unified Apostol-Bernoulli-Euler polynomials ${}_H\beta_n(\rho, \sigma, \tau, \xi)$:*

$$\left[\frac{2-\xi+\frac{1}{2}\xi t}{\tau e^t + (1-\xi)}\right] e^{\rho t + \sigma t^2} = \sum_{n=0}^{\infty} {}_H\beta_n(\rho, \sigma, \tau, \xi)\frac{t^n}{n!}, \tag{2.1}$$

$$\begin{cases} \left| ln\left(\frac{\tau}{1-\xi} + t\right)\right| < 2\pi, & \text{for} \quad 0 \le \xi < 1 \\ \\ \left| ln\left(\frac{\tau}{\xi-1} + t\right)\right| < \pi, & \text{otherwise.} \end{cases}$$

Parameters	**Generating functions**	**Polynomials**
$\xi = 2,\ \tau = 1$	$\frac{t}{e^t-1}e^{\rho t+\sigma t^2} = \sum_{n=0}^{\infty} {}_H\mathcal{B}_n(\rho,\sigma)\frac{t^n}{n!}$	Hermit-Bernoulli polynomials
$\xi = 0, \tau = 1$	$\frac{2}{e^t+1}e^{\rho t+\sigma t^2} = \sum_{n=0}^{\infty} {}_H\mathcal{E}_n(\rho,\sigma)\frac{t^n}{n!}$	Hermite-Euler polynomials
$\xi = 2, \tau = 1, \sigma = 0$	$\frac{t}{e^t-1}e^{\rho t} = \sum_{n=0}^{\infty} \mathcal{B}_n(\rho)\frac{t^n}{n!} \qquad \lvert t\rvert < 2\pi$	Bernoulli polynomials
$\xi = 0, \tau = 1, \sigma = 0$	$\frac{2}{e^t+1}e^{\rho t} = \sum_{n=0}^{\infty} \mathcal{E}_n(\rho)\frac{t^n}{n!} \qquad \lvert t\rvert < \pi$	Euler polynomials
$\xi = 2$	$\frac{t}{\tau e^t-1}e^{\rho t+\sigma t^2} = \sum_{n=0}^{\infty} {}_H\mathcal{B}_n(\rho,\sigma,\tau)\frac{t^n}{n!}$	Apostol Hermit-Bernoulli polynomials
$\xi = 0$	$\frac{2}{\tau e^t+1}e^{\rho t+\sigma t^2} = \sum_{n=0}^{\infty} {}_H\mathcal{E}_n(\rho,\sigma,\tau)\frac{t^n}{n!}$	Apostol Hermite-Euler polynomials
$\xi = 2, \sigma = 0$	$\frac{t}{\tau e^t-1}e^{\rho t} = \sum_{n=0}^{\infty} \mathcal{B}_n(\rho,\tau)\frac{t^n}{n!} \qquad \lvert t+ln\tau\rvert < 2\pi$	Apostol-Bernoulli polynomials
$\xi = 0, \sigma = 0$	$\frac{2}{\tau e^t+1}e^{\rho t} = \sum_{n=0}^{\infty} \mathcal{E}_n(\rho,\tau)\frac{t^n}{n!} \qquad \lvert t+ln\tau\rvert < \pi$	Apostol-Euler polynomials

We now list particular cases obtained by assigning specific values to the parameters in (2.1).

Determinant representations provide an elegant matrix-based framework for Bernoulli, Euler and Genocchi polynomials, facilitating derivation of their key properties. This approach, initiated by Costabile, F. A. and Longo, E. [7] for Appell polynomials, was extended Euler-Genocchi and Bernoulli-Euler variants by Belbachir, H., Hadj-Brahim, S. and Rachidi, M. [4, 5], yielding explicit formulas and combinatorial identities. In this section, we introduce the determinantal representation of Hermite-based unified Apostol-Bernoulli-Euler polynomials, which generalize some results of Belbachir, H., Djemmada, Y. and Hadj-Brahim, S. [6].

The following power series yield the Hermite-Bernoulli polynomials of Apostol type and the Hermite-Euler polynomials of Apostol type.

$$\frac{t}{\tau e^t-1}e^{\rho t+\sigma t^2}=\sum_{n=0}^{\infty}{}_H\mathcal{B}_n(\rho,\sigma,\tau)\frac{t^n}{n!}\qquad(|t+ln\tau|<2\pi,\tau\in\mathbb{R}_+^*),\tag{2.2}$$

$$\frac{2}{\tau e^t+1}e^{\rho t+\sigma t^2}=\sum_{n=0}^{\infty}{}_H\mathcal{E}_n(\rho,\sigma,\tau)\frac{t^n}{n!}\qquad(|t+ln\tau|<\pi,\tau\in\mathbb{R}_+^*)\tag{2.3}$$

suppose

$$D(\rho,\sigma,\tau,t)=\frac{t}{\tau e^t-1}e^{\rho t+\sigma t^2}\times\frac{2}{\tau e^t+1}e^{\rho t+\sigma t^2}$$
$$D(\rho,\sigma,\tau,t)=\frac{2t}{\tau^2e^{2t}-1}e^{2\rho t+2\sigma t^2},$$

considering the expressions on the right-hand side of equations (2.2) and (2.3), a straightforward calculation yields

$$\tau^2D(\rho+1,\sigma,\tau,t)-D(\rho,\sigma,\tau,t)$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}[\tau^2{}_H\mathcal{B}_{n-k}(\rho+1,\sigma,\tau){}_H\mathcal{E}_k(\rho+1,\sigma,\tau)-{}_H\mathcal{B}_{n-k}(\rho,\sigma,\tau){}_H\mathcal{E}_k(\rho,\sigma,\tau)]\frac{t^n}{n!}.\tag{2.4}$$

On the other hand, we have

$$\tau^2D(\rho+1,\sigma,\tau,t)-D(\rho,\sigma,\tau,t)$$
$$=\tau^2\frac{2t}{\tau^2e^{2t}-1}e^{2(\rho+1)t+2\sigma t^2}-\frac{2t}{\tau^2e^{2t}-1}e^{2\rho t+2\sigma t^2}$$

$$\tau^2D(\rho+1,\sigma,\tau,t)-D(\rho,\sigma,\tau,t)=2\sum_{n=1}^{\infty}n\mathcal{H}_{n-1}(2\rho,2\sigma)\frac{t^n}{n!}.\tag{2.5}$$

By comparing the two equations (2.4) and (2.5), we formulate the next result.

Theorem 2.1. *Let ρ, σ are real numbers ($\rho, \sigma \in \mathbb{R}$), and n an integer ($n \in \mathbb{Z}$). Then*

$$\mathcal{H}_n(\rho, \sigma) = \sum_{k=0}^{n+1} \omega_{n,k} \times \eta_{n+1-k,k}(\rho, \sigma, \tau), \tag{2.6}$$

where

$$\omega_{n,k} = \frac{1}{2^{n-k+1}(n+1)} \binom{n+1}{k} \text{ and } \eta_{n,k}(\rho, \sigma, \tau) = \begin{vmatrix} \tau_H\mathcal{B}_n(\rho+1, \sigma, \tau) & {}_H\mathcal{E}_k(\rho, \sigma, \tau) \\ {}_H\mathcal{B}_n(\rho, \sigma, \tau) & \tau_H\mathcal{E}_k(\rho+1, \sigma, \tau) \end{vmatrix}.$$

Specifically, if we choose $\tau = 1$ in equation (2.6), we obtain the following result in terms of Hermite-Bernoulli and the Hermite-Euler polynomials.

Corollary 2.7. *Let ρ, σ be real numbers and $n \geq 0$, we have*

$$\mathcal{H}_n(\rho, \sigma) = \frac{1}{(n+1)} \sum_{k=0}^{n+1} \frac{1}{2^{n-k+1}} \binom{n+1}{k} \begin{vmatrix} {}_H\mathcal{B}_{n-(k-1)}(\rho+1, \sigma) & {}_H\mathcal{E}_k(\rho, \sigma) \\ {}_H\mathcal{B}_{n-(k-1)}(\rho, \sigma) & {}_H\mathcal{E}_k(\rho+1, \sigma) \end{vmatrix}. \tag{2.7}$$

Corollary 2.2. *For $\sigma = 0$, (2.6) reduces to the well-known result of Belbachir, H., Djemmada, Y. and Hadj-Brahim, S. (see* [6]*, p.2(3)):*

$$\rho^n = \sum_{k=0}^{n+1} \omega_{n,k} \times \eta_{n+1-k,k}(\rho, \tau)$$

$$\rho^n = \frac{1}{2^{n+1}(n+1)} \sum_{k=0}^{n+1} \binom{n+1}{k} \begin{vmatrix} \mathcal{B}_{n-(k-1)}(\rho+1, \tau) & \mathcal{E}_k(\rho, \tau) \\ \mathcal{B}_{n-(k-1)}(\rho, \tau) & \mathcal{E}_k(\rho+1, \tau) \end{vmatrix}.$$

3. Some Explicit Formulas

In this section, we present multiple explicit representations for the Hermit-based unified Apostol-Bernoulli-Euler polynomials. These formulas provide a direct means of computing the polynomials without relaying solely on recursive relations or generating function. Furthermore, these explicit results serve as the foundation for exploring further identities and analytical properties within the broader framework of hybrid polynomial sets.

Theorem 3.1. *Let $n \geq 0$ and $\xi \neq 1$, we have*

$${}_H\beta_n(\rho, \sigma, \tau, \xi) = \frac{1}{(1-\xi)} \left[\left(1 - \frac{\xi}{2}\right) {}_H\mathcal{E}_n\left(\rho, \sigma, \frac{\tau}{1-\xi}\right) - \frac{\xi}{2} {}_H\mathcal{B}_n\left(\rho, \sigma, \frac{\tau}{\xi - 1}\right) \right]. \tag{3.1}$$

Proof. Using the generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_H\beta_n(\rho,\sigma,\tau,\xi)\frac{t^n}{n!} = \left[\frac{2-\xi+\frac{1}{2}\xi t}{\tau e^t+(1-\xi)}\right]e^{\rho t+\sigma t^2}$$

$$= \frac{2-\xi}{2(1-\xi)}\left[\frac{2}{\frac{\tau e^t}{1-\xi}+1}\right]e^{\rho t+\sigma t^2} + \frac{\xi}{2(\xi-1)}\left[\frac{t}{\frac{\tau e^t}{\xi-1}-1}\right]e^{\rho t+\sigma t^2}$$

$$= \frac{1}{(1-\xi)}\sum_{n=0}^{\infty}\left[\left(1-\frac{\xi}{2}\right){}_H\mathcal{E}_n\left(\rho,\sigma,\frac{\tau}{1-\xi}\right) - \frac{\xi}{2}{}_H\mathcal{B}_n\left(\rho,\sigma,\frac{\tau}{\xi-1}\right)\right]\frac{t^n}{n!}.$$

Theorem (3.1), is obtained by equating the coefficients of t^n on both sides.

Corollary 3.1. *For $\sigma = 0$, (3.1) reduces to a result of Belbachir, H., Djemmada, Y. and Hadj-Brahim, S.* [6].

$$\mathcal{B}_n(\rho,\tau,\xi) = \frac{1}{(1-\xi)}\left[\left(1-\frac{\xi}{2}\right)\mathcal{E}_n\left(\rho,\frac{\tau}{1-\xi}\right) - \frac{\xi}{2}\mathcal{B}_n\left(\rho,\frac{\tau}{\xi-1}\right)\right]. \tag{3.2}$$

Theorem 3.2. *For $\tau \in \mathbb{R}_+^*$ and $\xi \neq 1$, it holds that*

$${}_H\beta_n(\rho,\sigma,\tau,\xi) = \frac{1}{2(1-\xi)}\left[(2-\xi)\,{}_H\mathcal{E}_n\left(\rho,\sigma,\frac{\tau}{1-\xi}\right) + \frac{n\xi}{2}{}_H\mathcal{E}_{n-1}\left(\rho,\sigma,\frac{\tau}{1-\xi}\right)\right]. \tag{3.3}$$

Proof. Using the generating function (2.1), we get

$$\sum_{n=0}^{\infty} {}_H\beta_n(\rho,\sigma,\tau,\xi)\frac{t^n}{n!} = \left[\frac{2-\xi+\frac{1}{2}\xi t}{\tau e^t+(1-\xi)}\right]e^{\rho t+\sigma t^2}$$

$$= \frac{2-\xi}{2(1-\xi)}\left[\frac{2}{\frac{\tau e^t}{1-\xi}+1}\right]e^{\rho t+\sigma t^2} + \frac{\xi t}{4(1-\xi)}\left[\frac{2}{\frac{\tau e^t}{1-\xi}+1}\right]e^{\rho t+\sigma t^2}$$

$$= \frac{1}{2(1-\xi)}\left[(2-\xi)\sum_{n=0}^{\infty}{}_H\mathcal{E}_n\left(\rho,\sigma,\frac{\tau}{1-\xi}\right)\frac{t^n}{n!} + \frac{\xi}{2}\sum_{n=0}^{\infty}{}_H\mathcal{E}_n\left(\rho,\sigma,\frac{\tau}{1-\xi}\right)\frac{t^{n+1}}{n!}\right]$$

$$= \frac{1}{2(1-\xi)}\sum_{n=1}^{\infty}\left[(2-\xi)\quad {}_H\mathcal{E}_n\left(\rho,\sigma,\frac{\tau}{1-\xi}\right) + \frac{\xi n}{2}\quad {}_H\mathcal{E}_{n-1}\left(\rho,\sigma,\frac{\tau}{1-\xi}\right)\right]\frac{t^n}{n!}.$$

Result (3.3) is obtained by equating the coefficients of t^n on both sides.

Corollary 3.2. *For $\sigma = 0$, (3.3) reduces to the result of Belbachir, H., Djemmada, Y. and Hadj-Brahim, S.* [6].

$$\mathcal{B}_n(\rho,\tau,\xi) = \frac{1}{2(\xi-1)}\left[(\xi-2)\mathcal{E}_n\left(\rho,\frac{\tau}{1-\xi}\right) - \frac{\xi n}{2}\mathcal{E}_{n-1}\left(\rho,\frac{\tau}{1-\xi}\right)\right]. \tag{3.4}$$

Theorem 3.3. *If n is positive integer ($n \in \mathbb{Z} \setminus \{0\}$), then*

$$ {}_H\beta_n(\rho+u,\sigma+v,\tau,\xi) = \sum_{k=0}^{n} \binom{n}{k} {}_H\beta_n(\rho,\sigma,\tau,\xi) \quad \mathcal{H}_{n-k}(u,v). \tag{3.5}$$

Proof. Exploiting the generating function (2.1) by replacing ρ with $\rho+u$ and σ with $\sigma+v$, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_H\beta_n(\rho+u,\sigma+v,\tau,\xi)\frac{t^n}{n!} = \left[\frac{2-\xi+\frac{1}{2}\xi t}{\tau e^t+(1-\xi)}\right] e^{(\rho+u)t+(\sigma+v)t^2}\\
&= \left[\frac{2-\xi+\frac{1}{2}\xi t}{\tau e^t+(1-\xi)}\right] e^{\rho t+\sigma t^2} e^{ut+vt^2}\\
&= \sum_{k=0}^{\infty} {}_H\beta_k(\rho,\sigma,\tau,\xi)\frac{t^k}{k!} \quad \sum_{n=0}^{\infty} \mathcal{H}_n(u,v)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} {}_H\beta_k(\rho,\sigma,\tau,\xi)\mathcal{H}_{n-k}(u,v)\frac{t^n}{n!}.
\end{aligned}$$

Theorem (3.3) is obtained by equating the coefficients of t^n on both sides.

Corollary 3.3. *For $v=0$ and $u=z$, Theorem (3.3) reduces as:*

$$ {}_H\beta_n(\rho+\sigma,z,\tau,\xi) = \sum_{k=0}^{n} \binom{n}{k} {}_H\beta_k(\rho,z,\tau,\xi)\sigma^{n-k}. \tag{3.6}$$

Proof. Replacing ρ by $\rho+\sigma$ on the generating function (2.1), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_H\beta_n(\rho+\sigma,z,\tau,\xi)\frac{t^n}{n!} = \left[\frac{2-\xi+\frac{1}{2}\xi t}{\tau e^t+(1-\xi)}\right] e^{(\rho+\sigma)t+zt^2}\\
&= \left[\frac{2-\xi+\frac{1}{2}\xi t}{\tau e^t+(1-\xi)}\right] e^{(\rho t+zt^2)} \quad e^{\sigma t}\\
&= \sum_{k=0}^{\infty} {}_H\beta_k(\rho,z,\tau,\xi) \quad \sum_{n=0}^{\infty} \frac{\sigma^n t^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} {}_H\beta_k(\rho,z,\tau,\xi)\sigma^{n-k}\frac{t^n}{n!}.
\end{aligned}$$

Equating the coefficients of t^n on both sides, we obtain (3.6).

Corollary 3.4. *In particular if we substitute $\sigma = -\rho$ and z=0 in equation (3.6), we have*

$$\begin{aligned} {}_H\beta_n(0,0,\tau,\xi) &= \sum_{k=0}^{n} \binom{n}{k} {}_H\beta_k(\rho,\tau,\xi)(-\rho)^{n-k} \\ {}_H\beta_n(\tau,\xi) &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} {}_H\beta_k(\rho,\tau,\xi) \quad \rho^{n-k}. \end{aligned} \tag{3.7}$$

Which are the Hermite based unified Apostol-Bernoulli-Euler numbers.

Theorem 3.4. *For $\tau \in \mathbb{R}_+^*$, $\xi \in \mathbb{R}_+ - \{1\}$ and $n \geq 1$, we have*

$$\begin{aligned} &{}_H\beta_{n+1}(\rho,\sigma,\tau,\xi) - \rho \ {}_H\beta_n(\rho,\sigma,\tau,\xi) - 2\sigma n \ {}_H\beta_{n-1}(\rho,\sigma,\tau,\xi) \\ &= \frac{1}{2(1-\xi)} \left[\frac{\xi}{2} \mathcal{E}_n \left(\rho, \sigma, \frac{\tau}{(1-\xi)}\right) - \tau \sum_{k=0}^{\infty} \binom{n}{k} \mathcal{E}_k \left(\frac{\tau}{1-\xi}\right) {}_H\beta_{n-k}(\rho+1,\sigma,\tau,\xi) \right]. \end{aligned} \tag{3.8}$$

Proof. Using the generating function (2.1), we get

$$\left[\frac{2-\xi+\frac{\xi t}{2}}{\tau e^t + (1-\xi)}\right] e^{\rho t + \sigma t^2} = \sum_{n=0}^{\infty} {}_H\beta_n(\rho,\sigma,\tau,\xi) \frac{t^n}{n!}$$

differentiate both sides with respect to t, we get

$$\begin{aligned} &\frac{d}{dt} \sum_{n=0}^{\infty} \beta_n(\rho,\sigma,\tau,\xi) \frac{t^n}{n!} = \frac{d}{dt} \left\{ \left[\frac{2-\xi+\frac{\xi t}{2}}{\tau e^t + (1-\xi)}\right] e^{\rho t + \sigma t^2} \right\} \\ &= \frac{\xi}{4(1-\xi)} \left[\frac{2}{\frac{\tau e^t}{1-\xi} + 1}\right] e^{\rho t + \sigma t^2} - \tau \left[\frac{2-\xi+\frac{1}{2}\xi t}{\tau e^t + (1-\xi)}\right] e^{(\rho+1)t + \sigma t^2} \times \frac{1}{\tau e^t + (1-\xi)} \\ &+ \rho \left[\frac{2-\xi+\frac{1}{2}\xi t}{\tau e^t + (1-\xi)}\right] e^{\rho t + \sigma t^2} + 2\sigma t \left[\frac{2-\xi+\frac{1}{2}\xi t}{\tau e^t + (1-\xi)}\right] e^{\rho t + \sigma t^2} \\ &= \frac{\xi}{4(1-\xi)} \sum_{n=0}^{\infty} {}_H\mathcal{E}_n \left(\rho, \sigma, \frac{\tau}{1-\xi}\right) \frac{t^n}{n!} - \tau \quad \sum_{n=0}^{\infty} {}_H\beta_n(\rho+1,\sigma,\tau,\xi) \frac{t^n}{n!} \\ &\times \frac{1}{2(1-\xi)} \left[\frac{2}{\frac{\tau}{1-\xi} e^t + 1}\right] + \rho \sum_{n=0}^{\infty} {}_H\beta_n(\rho,\sigma,\tau,\xi) \frac{t^n}{n!} + 2\sigma \sum_{n=0}^{\infty} {}_H\beta_n(\rho,\sigma,\tau,\xi) \frac{t^{n+1}}{n!} \\ &\sum_{n=0}^{\infty} {}_H\beta_{n+1}(\rho,\sigma,\tau,\xi) \frac{t^n}{n!} \end{aligned}$$

$$=\sum_{n=0}^{\infty}\frac{1}{2(1-\xi)}\left\{\frac{\xi}{2}{}_H\mathcal{E}_n(\rho,\sigma,\frac{\tau}{1-\xi})-\tau\sum_{k=0}^{n}\binom{n}{k}{}_H\beta_{n-k}(\rho+1,\sigma,\tau,\xi)\mathcal{E}_k\left(\frac{\tau}{1-\xi}\right)\right\}$$

$$\frac{t^n}{n!}+\rho\sum_{n=0}^{\infty}{}_H\beta_n(\rho,\sigma,\tau,\xi)\frac{t^n}{n!}+\sum_{n=1}^{\infty}2\sigma n{}_H\beta_{n-1}(\rho,\sigma,\tau,\xi)\frac{t^n}{n!}.$$

Comparing the coefficients of t^n on both sides, we get

$$\begin{aligned}&{}_H\beta_{n+1}(\rho,\sigma,\tau,\xi)-\rho\ {}_H\beta_n(\rho,\sigma,\tau,\xi)-2\sigma n\ {}_H\beta_{n-1}(\rho,\sigma,\tau,\xi)\\&=\frac{1}{2(1-\xi)}\left[\frac{\xi}{2}\mathcal{E}_n\left(\rho,\sigma,\frac{\tau}{(1-\xi)}\right)-\tau\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\tau}{1-\xi}\right){}_H\beta_{n-k}(\rho+1,\sigma,\tau,\xi)\right].\end{aligned}$$

Theorem 3.5. *For Hermite based unified Apostol-Bernoulli-Euler polynomials, the implicit summing formula is valid.*

$$_H\beta_{k+l}(z,\sigma,\tau,\xi)=\sum_{n,m=0}^{k,l}\binom{k}{n}\binom{l}{m}(z-\rho)^{n+m}{}_H\beta_{k+l-n-m}(\rho,\sigma,\tau,\xi). \tag{3.9}$$

Proof. We rewrite the generating function (2.1) as:

$$\left[\frac{2-\xi+\frac{1}{2}\xi(t+u)}{\tau e^{(t+u)}+(1-\xi)}\right]e^{\rho(t+u)+\sigma(t+u)^2}=\sum_{k,l=0}^{\infty}{}_H\beta_{k+l}(\rho,\sigma,\tau,\xi)\frac{t^k}{k!}\frac{u^l}{l!}$$

$$\left[\frac{2-\xi+\frac{1}{2}\xi(t+u)}{\tau e^{(t+u)}+(1-\xi)}\right]e^{\sigma(t+u)^2}=e^{-\rho(t+u)}\sum_{k,l=0}^{\infty}{}_H\beta_{k+l}(\rho,\sigma,\tau,\xi)\frac{t^k}{k!}\frac{u^l}{l!}$$

multiply both sides by $e^{z(t+u)}$, we obtain

$$\left[\frac{2-\xi+\frac{1}{2}\xi(t+u)}{\tau e^{(t+u)}+(1-\xi)}\right]e^{z(t+u)+\sigma(t+u)^2}=e^{(z-\rho)(t+u)}\sum_{k,l=0}^{\infty}{}_H\beta_{k+l}(\rho,\sigma,\tau,\xi)\frac{t^k}{k!}\frac{u^l}{l!}$$

$$\sum_{k,l=0}^{\infty}{}_H\beta_{k+l}(z,\sigma,\tau,\xi)\frac{t^k}{k!}\frac{u^l}{l!}=\sum_{N=0}^{\infty}\frac{[(z-\rho)(t+u)]^N}{N!}\sum_{k,l=0}^{\infty}{}_H\beta_{k+l}(\rho,\sigma,\tau,\xi)\frac{t^k}{k!}\frac{u^l}{l!}$$

applying the formula ([18], p.52(2))

$$\sum_{N=0}^{\infty}f(N)\frac{(\rho+\sigma)^N}{N!}=\sum_{n,m=0}^{\infty}f(n+m)\frac{\rho^n}{n!}\frac{\sigma^m}{m!}.$$

$$\sum_{k,l=0}^{\infty} {}_H\beta_{k+l}(z,\sigma,\tau,\xi)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{n,m=0}^{\infty}(z-\rho)^{n+m}\frac{t^n}{n!}\frac{u^m}{m!}\sum_{k,l=0}^{\infty} {}_H\beta_{k+l}(\rho,\sigma,\tau,\xi)\frac{t^k}{k!}\frac{u^l}{l!}$$

$$= \sum_{k,l=0}^{\infty}\sum_{n,m=0}^{k,l}\binom{k}{n}\binom{l}{m}(z-\rho)^{n+m}{}_H\beta_{k+l-n-m}(\rho,\sigma,\tau,\xi)\frac{t^k}{k!}\frac{u^l}{l!}.$$

Equating the coefficients of same powers of t and u, we get the desired result.

Corollary 3.5. *For $\xi = 2$ and $\tau = 1$, (3.9) reduces to well-known result of Pathan, M. A. and Khan, W. A. ([16], p.688(3.12)):*

$${}_H\beta_{k+l}(z,\sigma,1,2) = \sum_{n,m=0}^{k,l}\binom{k}{n}\binom{l}{m}(z-\rho)^{n+m}{}_H\beta_{k+l-n-m}(\rho,\sigma,1,2),$$

$${}_H\mathcal{B}_{k+l}(z,\sigma) = \sum_{n,m=0}^{k,l}\binom{k}{n}\binom{l}{m}(z-\rho)^{n+m}{}_H\mathcal{B}_{k+l-n-m}(\rho,\sigma). \tag{3.10}$$

Corollary 3.6. *Substitute $\tau = 1$ and $\xi = 0$ in equation (3.9), we get*

$${}_H\beta_{k+l}(z,\sigma,1,0) = \sum_{n,m=0}^{k,l}\binom{k}{n}\binom{l}{m}(z-\rho)^{n+m}{}_H\beta_{k+l-n-m}(\rho,\sigma,1,0)$$

$${}_H\mathcal{E}_{k+l}(z,\sigma) = \sum_{n,m=0}^{k,l}\binom{k}{n}\binom{l}{m}(z-\rho)^{n+m}{}_H\mathcal{E}_{k+l-n-m}(\rho,\sigma) \tag{3.11}$$

4. General Symmetric Identities

Utilizing the generating function defined in (2.1), we drive several comprehensive symmetric identities for the Hermite-based unified Apostol-Bernoulli-Euler polynomials. These identities provide a generalized approach to understanding the interplay between the variables and parameters within the unified class. The theorems presented herein serve as essential tools for simplifying complex functional equations and expanding the known properties of generalized Appell-type sequences.

Theorem 4.1. *The following identity is valid for all integers $a > 0$, $b > 0$ and $n \geq 0$:*

$$\begin{aligned}&\sum_{k=0}^{n} {}_H\beta_{n-k}(b\rho,b^2\sigma,\tau,\xi){}_H\beta_k(a\rho,a^2\sigma,\tau,\xi)\frac{a^{n-k}b^k}{k!(n-k)!}\\ &= \sum_{k=0}^{n} {}_H\beta_{n-k}(a\rho,a^2\sigma,\tau,\xi){}_H\beta_k(b\rho,b^2\sigma,\tau,\xi)\frac{b^{n-k}a^k}{k!(n-k)!}.\end{aligned} \tag{4.1}$$

Proof. Let the expression of g(t), is symmetric in a and b both, then

$$g(t) = \left[\frac{(2-\xi+\frac{1}{2}\xi at)(2-\xi+\frac{1}{2}\xi bt)}{(\tau e^{at}+(1-\xi))(\tau e^{bt}+(1-\xi))}\right][exp(ab\rho t+a^2b^2\sigma t^2)]^2$$

with the aid of (2.1), there are two ways we can expand g(t) into series

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} {}_H\beta_n(a\rho, a^2\sigma, \tau, \xi)\frac{(bt)^n}{n!}\sum_{k=0}^{\infty} {}_H\beta_k(b\rho, b^2\sigma, \tau, \xi)\frac{(at)^k}{k!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_H\beta_{n-k}(a\rho, a^2\sigma, \tau, \xi){}_H\beta_k(b\rho, b^2\sigma, \tau, \xi)\frac{b^{n-k}a^k}{k!(n-k)!}t^n. \end{aligned} \tag{4.2}$$

In the same way, we may demonstrate that

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} {}_H\beta_n(b\rho, b^2\sigma, \tau, \xi)\frac{(at)^n}{n!}\sum_{k=0}^{\infty} {}_H\beta_k(a\rho, a^2\sigma, \tau, \xi)\frac{(bt)^k}{k!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_H\beta_{n-k}(b\rho, b^2\sigma, \tau, \xi){}_H\beta_k(a\rho, a^2\sigma, \tau, \xi)\frac{a^{n-k}b^k}{k!(n-k)!}t^n. \end{aligned} \tag{4.3}$$

Comparing the equations (4.2) and (4.3), we get (4.1).

Corollary 4.1. *For $\tau = 1$, $\xi = 2$, theorem (4.1) reduces to a known result of Pathan, M. A. and Khan, W. A. (for $\alpha = 1, m = 1$) ([16], p.689(4.1)) :*

$$\sum_{k=0}^{n} {}_H\mathcal{B}_{n-k}(b\rho, b^2\sigma){}_H\mathcal{B}_k(a\rho, a^2\sigma)\frac{a^{n-k}b^k}{k!(n-k)!} = \sum_{k=0}^{n} {}_H\mathcal{B}_{n-k}(a\rho, a^2\sigma){}_H\mathcal{B}_k(b\rho, b^2\sigma)\frac{b^{n-k}a^k}{k!(n-k)!}. \tag{4.4}$$

Remark 4.1. *By setting b=1 in (4.4), it reduces to the result given by Pathan, M. A. and Khan, W. A.* [16]*:*

$$\sum_{k=0}^{n} {}_H\mathcal{B}_{n-k}(a\rho, a^2\sigma){}_H\mathcal{B}_k(\rho, \sigma)\frac{a^k}{k!(n-k)!} = \sum_{k=0}^{n} {}_H\mathcal{B}_{n-k}(\rho, \sigma){}_H\mathcal{B}_k(a\rho, a^2\sigma)\frac{a^{n-k}}{k!(n-k)!}.$$

Theorem 4.2. *The following identity holds true for all integers $a > 0$, $b > 0$ and $n \geq 0$:*

$$\sum_{k=0}^{n} {}_H\beta_{n-k}(b\rho, b^2z, \tau, \xi)\mathcal{B}_k(a\sigma, \tau, \xi)\frac{a^{n-k}b^k}{k!(n-k)!} = \sum_{k=0}^{n} {}_H\beta_{n-k}(a\rho, a^2z, \tau, \xi)\mathcal{B}_k(b\sigma, \tau, \xi)\frac{b^{n-k}a^k}{k!(n-k)!}. \tag{4.5}$$

Proof. The expression of g(t) is symmetric in both a and b, and using equations (2.1) and (1.12), we get

$$\begin{aligned} g(t) &= \left[\frac{(2-\xi+\frac{1}{2}\xi at)(2-\xi+\frac{1}{2}\xi bt)}{(\tau e^{at}+(1-\xi))(\tau e^{bt}+(1-\xi))}\right] e^{ab(\rho+\sigma)t+a^2b^2zt} \\ &= \left[\frac{2-\xi+\frac{1}{2}\xi at}{\tau e^{at}+(1-\xi)}\right] e^{ab\rho t+a^2b^2zt} \quad \left[\frac{2-\xi+\frac{1}{2}\xi bt}{\tau e^{bt}+(1-\xi)}\right] e^{ab\sigma t} \\ &= \sum_{n=0}^{\infty} {}_H\beta_n(b\rho, b^2z, \tau, \xi)\frac{(at)^n}{n!} \sum_{k=0}^{\infty} \mathcal{B}_k(a\sigma, \tau, \xi)\frac{(bt)^k}{k!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_H\beta_{n-k}(b\rho, b^2z, \tau, \xi)\mathcal{B}_k(a\sigma, \tau, \xi)\frac{a^{n-k}b^k}{k!(n-k)!}t^n \end{aligned} \tag{4.6}$$

also we can write g(t) as

$$g(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_H\beta_{n-k}(a\rho, a^2z, \tau, \xi)\mathcal{B}_k(b\sigma, \tau, \xi)\frac{b^{n-k}a^k}{k!(n-k)!}t^n. \tag{4.7}$$

Comparing the equations (4.6) and (4.7), we get the identity (4.5).

Corollary 4.2. *Using $\tau = 1$, $\xi = 2$ in equation (4.5), we obtain:*

$$\sum_{k=0}^{n} {}_H\mathcal{B}_{n-k}(a\rho, a^2z)\mathcal{B}_k(b\sigma)\frac{b^{n-k}a^k}{k!(n-k)!} = \sum_{k=0}^{n} {}_H\mathcal{B}_{n-k}(b\rho, b^2z)\mathcal{B}_k(a\sigma)\frac{a^{n-k}b^k}{k!(n-k)!}.$$

5. Conclusion

This paper explores Hermite based Apostol-Bernoulli-Euler polynomials, deriving implicit formulas, symmetric identities, and determinant forms. These polynomials, particularly in their generalized and multi-variable forms, have shown significant potential in solving partial differential equations prevalent in mathematical physics. By introducing special polynomials in two variables, we offer innovative analytical tools for addressing complex physical phenomena. Our findings enhance both the theoretical understanding and practical applications of these functions. Future research may extend these polynomials to higher dimensions, further contributing to the advancement of mathematical methods in physics and related fields.

Additionally, extending these polynomials to the q-calculus framework using q-exponentials could uncover new mixed functions, offering deeper insights into quantum theory and its applications. The two q-exponential functions are defined respectively as [10]:

$$e_q(\rho) = \sum_{n=0}^{\infty} \frac{\rho^n}{[n]_q!}, \qquad 0 < |q| < 1 \tag{5.1}$$

and

$$E_q(\rho) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} \rho^n}{[n]_q!}, \qquad 0 < |q| < 1. \tag{5.2}$$

Where

$$e_q(\rho) E_q(-\rho) = 1$$

The q-Hermit-based unified Apostol Bernoulli-Euler polynomials defined by the generating function:

$$\left[\frac{2 - \xi + \frac{1}{2}\xi t}{\tau e^t + (1 - \xi)}\right] e_q(\rho t) e_q(\sigma^2 t) = \sum_{n=0}^{\infty} {}_H\beta_{n,q}(\rho, \sigma, \tau, \xi) \frac{t^n}{[n]_q!}, \qquad 0 < |q| < 1. \tag{5.3}$$

Although we have introduced the generating functions for these q-type polynomials, a deeper exploration of their unique characteristics is deferred to future study. Investigating these properties remains a primary focus for the next stage of this research.

References

[1] Apostol, T. M., On the Lerch zeta function, Pacific J. Math., 1(2) (1951), 161-167.

[2] Appell, P. and Kampe de Fériet, J., Fonctions Hypergeometriques et Hyperspheriques: Polynomes d'Hermite, Gauthier-Villars, Paris, 1926.

[3] Bernoulli, J., Ars conjectandi, Impensis Thurnisiorum, fratrum, 1713.

[4] Belbachir, H., Hadj-Brahim, S. and Rachidi, M., On another approach for a family of Appell polynomials, Filomat, 32(12) (2018), 4155-4164.

[5] Belbachir, H., Hadj-Brahim, S. and Rachidi, M., Other approaches for generalized Bernoulli-Euler polynomials and beyond, Rend. Mat. Appl. 7, 44 (2023), 211-235.

[6] Belbachir, H., Djemmada, Y. and Hadj-Brahim, S., Unified Bernoulli-Euler polynomials of Apostol type, Indian J. Pure Appl. Math., 54(1) (2023), 76-83.

[7] Costabile, F. A. and Longo, E., An algebraic approach to Sheffer polynomial sequences, Integral Transforms Spec. Funct., 25(4), 295-311.

[8] Dattoli, G., Lorenzutta, S. and Cesarano, C., Finite sums and generalized forms of Bernoulli polynomials, Rend. Mat. Appl., 19(3), 385-391.

[9] Euler, L., Methodus generalis summandi progressiones, Comm. Acad. Sci. Petrop., (1738), 68–97.

[10] Gasper, G. and Rahman, M., Basic Hypergeometric series, Cambridge University Press, Cambridge, 2004.

[11] Luo, Q. M. and Srivastava H. M., Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials, J. Math. Anal. Appl., 308(1) (2005), 290-302.

[12] Luo, Q. M. and Srivastava H. M., Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials, Comput. Math. Appl., 51(3-4) (2006), 631-642.

[13] Luo, Q. M., Apostol-Euler polynomials of higher order and Gaussian hypergeometric function, Taiwanese J. Math., 10(4) (2006), 917-925.

[14] Ozden, H., Simsek Y. and Srivastava, H. M., A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials, Comput. math. appl., 60(10) (2010), 2779-2787.

[15] Pathan, M. A., A new class of generalized Hermite–Bernoulli polynomials, Georgian math. j., 19(3) (2012), 559-573.

[16] Pathan, M. A. and Khan W. A., Some implicit summation formulas and symmetric identities for the generalized Hermite-Bernoulli polynomials, Acta Uni. Apulensis, 39 (2014), 113-136.

[17] Pathan, M. A. and Khan, W. A., Some implicit summation formulas and symmetric identities for the generalized Hermite-Euler polynomials, East-West J. Math., 16(1) (2014), 92-109.

[18] Srivastava, H. M. and Manocha, H. L., A treatise on generating functions, Ellis Horwood Limited, Halsted Press: a division of John Wily and Sons, New York, 1984.